LAPLACE DOMAIN ANALYSIS OF THE COLLINEAR VEHICLE-TO-VEHICLE COLLISION UNDER LINEAR FORCE-DEFLECTION CONSTRAINTS COUPLED WITH NET EXTERNALLY APPLIED LOADS

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ABSTRACT

The focus of the subject study was on the development of the closed-form solutions for displacement, velocity and acceleration, based upon the utilization of the Laplace transform, experienced by each collision partner involved in a collinear collision under the constraints of linearity in the force-deflection response during closure and separation and while subject to any number of net externally applied loads for which an analytic Laplace transform was determinable. Starting from the basic expression of Newton's Second Law, the coupled equations of motion were developed in standard matrix-vector form by the introduction of the definitions of structural deflections in terms of the displacements of the center of mass of each collision partners and the massless common collision interface. The solution for the equations of motion was determined by applying the Laplace Transform and determining the solution for the dynamic stiffness matrix and transfer function by means of an Eigendecomposition. The closed-form analytic for the Laplace domain displacement was readily amenable to the inverse Laplace Transform and thereby provided a closed-form analytic solution for displacement in the time domain. The first and second time derivatives of this solution provided the closed form solution for velocity and acceleration, respectively. The reduced forms of each of these three equations, addressing the specific limits of the achievement of common velocity at the terminus of closure and the achievement of zero acceleration due to the collision force at the terminus of separation, were then developed.

The use of the residue theorem, instead of a partial fraction expansion, for the evaluation of the transfer function, coupled with the reduction in the complexity of the general problem secondary to both the complex conjugate nature of the solutions for the Eigenvalue problem and the reduction of the same to only complex roots for the solution to the characteristic polynomial of the dynamic stiffness matrix for the undamped problem, revealed a solution set comprised of a rigid body mode and a solution based upon the circular frequency of the effective system mass and stiffness. Depending on the nature of the net externally applied forces, the solution for the time of terminus of closure could be determined from basic trigonometric relationships or from equating the velocities of the collision partners in the Laplace domain, solving for the Laplace variable and then performing an inverse Laplace transform to obtain the solution in the time domain. The terminus of the separation could be solved for in a similar manner through the use of the acceleration of either collision partner.

INTRODUCTION

Collinear vehicle-to-vehicle collisions, either in full actuality or reductive in the case of minimal vehicle rotation or lateral translation during the collision phase, thereby reducing the collision phase to that of global uniaxiality, represent a common situation encountered in the field of motor vehicle accident reconstruction. The analysis of such collisions, as with a substantial number of vehicle-to-vehicle collisions, is predicated upon the treatment of the collision partners, during the duration of the collision, as being a single system subject solely to the collision force, generated internally, within the system. Forces applied external to the system (e.g. braking, acceleration, drag, etc.) are typically neglected as their magnitude is substantially smaller than that of the magnitude of the collision force. There are two general situations in which such an approach may lead to the substantive underestimation of the collision severity. The first case is when one or both of the collision partners is interacting with at least one additional partner during the collision phase. The second is in the case of exceedingly low velocity collisions. There are a number of empirical studies within the scientific literature that detail the collision response for the latter case [1-5] or for relatable events [2, 6-8].

From the modeling perspective, the focus of the extant scientific literature has been within the context of modeling the effects of brake application during the collision phase. Emori and Horiguchi [1] and Siegmund et al. [2] modeled the effects of braking by

adding impulse terms to the post-collision side of the one dimensional conservation of linear momentum equation. This was combined with the definition of the coefficient of restitution to derive paired equations for the change in velocity of each collision partner in relation to the velocity of the striking vehicle under the case of the struck vehicle being at rest at the start of closure. Anderson et al. [4] expanded this formulation in terms of closing speed. Mastandrea and Vangi [9] approached the problem by separating the closure and separation phases. The separation force-deflection response was modeled as being bilinear with the inflection point being defined by the collision force magnitude corresponding to the collision force magnitude on loading associated with the linear bumper stiffness. The authors implemented a discretized form of the equations of motion and braking force, over each sequential time step, was taken as the average value between the time step. The cited literature has focused on the use of time domain techniques for modeling the collinear vehicle-to-vehicle problem, under global uniaxial constraints, and with net externally applied forces present.

The objective of the subject study was to apply an integral transform method, specifically the Laplace transform method, to the subject problem, under strict linear force-deflection modeling constraints.

THEORY

Collinearity with global uniaxiallity greatly simplifies the collision problem by reducing the rotation matrices associated with orientation to identity matrices and by reducing the degrees of freedom for each collision partner to unity. We consider the case of an aligned collinear collision between two collision partners. For ease of reference, the collision partners and their relevant parameters are referenced using the subscript notation of one and two, respectively. Two limiting factors are emplaced in regards to the subject theoretical development. The first is that the collision force is a function of the structural deflection of the collision partners. The second is that scope of externally applied unbalanced forces applied to each collision partner are either constant or temporally varying without the explicit inclusion of being functions of displacement. Terms in boldface type represent vector quantities.

The collision is temporally manifested by the start of closure, denoted as $t = t_o$, the terminus of closure/start of separation, denoted as $t = t_c$ and the terminus of separation, denoted as $t = t_s$. The structural deflection experienced by the contacted regions of the first and second collision partner, are denoted as $\boldsymbol{\delta}_1(t)$ and $\boldsymbol{\delta}_2(t)$, respectively. The peak deflection experienced by each collision partner, under the modeling limits, occurs at $t = t_c$ and is denoted as $\boldsymbol{\delta}_1(t_c) = \boldsymbol{\delta}_{1d}$ and $\boldsymbol{\delta}_2(t_c) = \boldsymbol{\delta}_{2d}$, respectively. The residual deformation (i.e. crush – a purely static parameter) for each collision partner, under the modeling limits, occurs at $t = t_s$ and is denoted as $\boldsymbol{\delta}_1(t_s) = \boldsymbol{\delta}_{1s}$ and $\boldsymbol{\delta}_2(t_s) = \boldsymbol{\delta}_{2s}$, respectively. The center of mass displacement experienced by each collision partner, measured with respect to an inertial frame of reference, is denoted by $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, respectively. The displacement of the common collision interface between the two collision partners, measured with respect to an inertial frame of reference, is denoted as $\mathbf{u}_{IF}(t)$. The single and double dot notation is used for velocity and acceleration, respectively. The collision force is denoted as $\mathbf{F}_c(t)$ and has a peak magnitude $|\mathbf{F}_{cp}|$ at $t = t_c$ and zero magnitude at $t = t_s$. Figure 1 depicts the collinear collision under consideration.

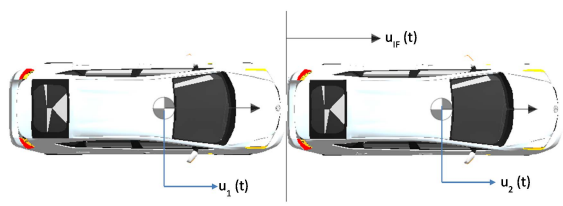

Figure 1. Collinear collision under consideration

Equations of motion for the collinear collision

The second order differential equation of motion for each collision partner can be written through the direct implementation of Newton's Second Law [10].

$$m_1\ddot{\mathbf{u}}_1(t) = -\mathbf{F}_c(t) + \mathbf{F}_{1ext} \qquad (1).$$

$$m_2\ddot{\mathbf{u}}_2(t) = \mathbf{F}_c(t) + \mathbf{F}_{2ext} \qquad (2).$$

As noted previously, the form of the externally applied unbalanced forces on either collision partner are in the form of a constant force $F_{jext} = F_{jconstant}$, a time varying force $F_{jext} = F_{jext}(t)$ or a combination of the two.

Linear loading and unloading models

The simplest method for modeling the force-deflection response is that of a linear relationship between force and deflection. In this case, each collision partner has a structural response defined as

k_1 and k_2, which differ from each other during closure (loading) and separation (unloading) and which differ for each collision partner during loading and unloading. The limitations of the linear-linear model have been detailed elsewhere [11-12]. We consider the time parametric force deflection response between time $t = t_a$ and time $t = t_b$ such that $t_b > t_a$. Figure 2 depicts a plot of the magnitude of the collision force as a function of the deflection for each collision partner between t_a and t_b.

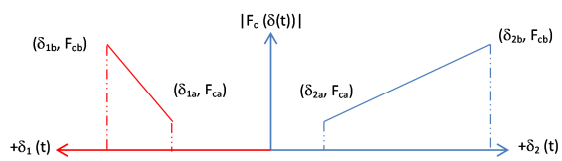

Figure 2. Magnitude of collision force plotted as a function of deflection for each collision partner for the linear force-deflection model.

The structural stiffness of each collision partner is simply the slope of the response between t_a and t_b. The force values shown in equation (3) represent the magnitude of the collision force and the subscripts "a" and "b" denote evaluation at $t = t_a$ and $t = t_b$ respectively.

$$k_1 = \frac{\mathbf{F}_{cb} - \mathbf{F}_{ca}}{\boldsymbol{\delta}_{1b} - \boldsymbol{\delta}_{1a}} \quad k_2 = \frac{\mathbf{F}_{cb} - \mathbf{F}_{ca}}{\boldsymbol{\delta}_{2b} - \boldsymbol{\delta}_{2a}} \qquad (3).$$

Each of the equations shown in (3) can be solved for $\mathbf{F}_{cb}$.

$$\mathbf{F}_{cb} = \mathbf{F}_{ca} + k_1\left(\boldsymbol{\delta}_{1b} - \boldsymbol{\delta}_{1a}\right) \quad \mathbf{F}_{cb} = \mathbf{F}_{ca} + k_2\left(\boldsymbol{\delta}_{2b} - \boldsymbol{\delta}_{2a}\right) \quad (4).$$

The magnitude of the collision force is equal, as determined from either vehicle, at all points in time during the collision. As a result, the two equations shown in (4) can be set equal to each other at t_b and $\mathbf{F}_{ca}$ can be eliminated from both sides of the equality.

$$k_1\left(\boldsymbol{\delta}_{1b} - \boldsymbol{\delta}_{1a}\right) = k_2\left(\boldsymbol{\delta}_{2b} - \boldsymbol{\delta}_{2a}\right) \qquad (5).$$

The structural deflections can be readily defined in terms of the center of mass displacements of each collision partner and in terms of the displacement of the common contact interface.

$$\begin{aligned} \boldsymbol{\delta}_1(t) &= \mathbf{u}_1(t) - \mathbf{u}_{IV}(t) \\ \boldsymbol{\delta}_2(t) &= \mathbf{u}_{IV}(t) - \mathbf{u}_2(t) \end{aligned} \qquad (6).$$

Defining the deflections for both collision partners at t_a and t_b, in terms of the displacements and substituting into equation (5) leads to a solution for $\mathbf{u}_{IFb}$.

$$\mathbf{u}_{IVb} = \mathbf{u}_{IVa} + \frac{\left(k_1\mathbf{u}_{1b} - k_1\mathbf{u}_{1a} + k_2\mathbf{u}_{2b} - k_2\mathbf{u}_{2a}\right)}{k_1 + k_2} \qquad (7).$$

A closed-form analytic solution for the displacement of the common collision interface at $t = t_b$ is one that is generally not developable in cases in which the force-deflection response is non-linear. Substitution of equation (7) into either of the equations shown in (4) results in a solution for the collision force in terms of the center of mass displacements at t_a and t_b.

$$\begin{aligned} \mathbf{F}_{cb} = \mathbf{F}_{ca} &+ \left(\frac{k_1k_2}{k_1+k_2}\right)\mathbf{u}_{1b} - \\ &\left(\frac{k_1k_2}{k_1+k_2}\right)\mathbf{u}_{2b} + \left(\frac{k_1k_2}{k_1+k_2}\right)\left(\mathbf{u}_{2a} - \mathbf{u}_{1a}\right) \end{aligned} \qquad (8).$$

Substitution of equation (8) into equations (1) and (2) provides for the form of the equations of motion for the specific linear force-deflection modeling methodology under consideration.

$$\begin{aligned} m_1\ddot{\mathbf{u}}_1 &+ \left(\frac{k_1k_2}{k_1+k_2}\right)\mathbf{u}_{1b} - \left(\frac{k_1k_2}{k_1+k_2}\right)\mathbf{u}_{2b} = \\ -\mathbf{F}_{ca} &- \left(\frac{k_1k_2}{k_1+k_2}\right)\left(\mathbf{u}_{2a} - \mathbf{u}_{1a}\right) + \mathbf{F}_{1ext} \end{aligned} \qquad (9).$$

$$\begin{aligned} m_2\ddot{\mathbf{u}}_2 &- \left(\frac{k_1k_2}{k_1+k_2}\right)\mathbf{u}_{1b} + \left(\frac{k_1k_2}{k_1+k_2}\right)\mathbf{u}_{2b} = \\ \mathbf{F}_{ca} &+ \left(\frac{k_1k_2}{k_1+k_2}\right)\left(\mathbf{u}_{2a} - \mathbf{u}_{1a}\right) + \mathbf{F}_{2ext} \end{aligned} \qquad (10).$$

The constant term, for each phase, defined by the quotient of the product of the individual stiffnesses to the sum of the stiffnesses, is termed the effective stiffness, k_{eff}. Equations (9) and (10) can be written in vector-matrix form.

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}_{ca} + \mathbf{F}_{\Delta ua} + \mathbf{F}_{ext} \qquad (11).$$

Where:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} k_{eff} & -k_{eff} \\ -k_{eff} & k_{eff} \end{bmatrix} \qquad (12).$$

$$\mathbf{F}_{ca} = \begin{bmatrix} -\mathbf{F}_{ca} \\ \mathbf{F}_{ca} \end{bmatrix} \quad \mathbf{F}_{\Delta ua} = \begin{bmatrix} -k_{eff}\left(\mathbf{u}_{2a} - \mathbf{u}_{1a}\right) \\ k_{eff}\left(\mathbf{u}_{2a} - \mathbf{u}_{1a}\right) \end{bmatrix}$$
$$\mathbf{F}_{ext} = \begin{bmatrix} \mathbf{F}_{1ext} \\ \mathbf{F}_{2ext} \end{bmatrix} \qquad (13).$$

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_1(t) \\ \mathbf{u}_2(t) \end{bmatrix} \quad \dot{\mathbf{u}}(t) = \begin{bmatrix} \dot{\mathbf{u}}_1(t) \\ \dot{\mathbf{u}}_2(t) \end{bmatrix}$$
$$\ddot{\mathbf{u}}(t) = \begin{bmatrix} \ddot{\mathbf{u}}_1(t) \\ \ddot{\mathbf{u}}_2(t) \end{bmatrix} \qquad (14).$$

For the analysis of the closure phase, the subscript "o" denoting the start of closure replaces the general subscript "a." The initial displacement conditions are generally taken as being zero. The column vectors $\mathbf{F}_{co}$ and $\mathbf{F}_{\Delta ua}$ are both $\mathbf{0}$. The initial velocities of the collision partners are defined as:

$$\dot{\mathbf{u}}_o = \begin{bmatrix} \dot{\mathbf{u}}_{1o} \\ \dot{\mathbf{u}}_{2o} \end{bmatrix} \qquad (15).$$

The effective stiffness during closure is derived from the loading stiffness values of both collision partners.

$$k_{eff} = \frac{k_{1load} k_{2load}}{k_{1load} + k_{2load}} \qquad (16).$$

The initial conditions for separation are the terminus conditions for closure. The constant valued forcing terms become:

$$\mathbf{F}_{ca} = \begin{bmatrix} -\mathbf{F}_{cp} \\ \mathbf{F}_{cp} \end{bmatrix} \quad \mathbf{F}_{\Delta ua} = \begin{bmatrix} -k_{eff}\left(\mathbf{u}_{2c} - \mathbf{u}_{1c}\right) \\ k_{eff}\left(\mathbf{u}_{2c} - \mathbf{u}_{1c}\right) \end{bmatrix} \qquad (17).$$

The effective stiffness in (17) is the effective stiffness for the unloading phase and is defined as:

$$k_{eff} = \frac{k_{1unload} k_{2unload}}{k_{1unload} + k_{2unload}} \qquad (18).$$

The initial conditions for the unload phase, using $t^* = t - t_c$, are:

$$\mathbf{u}_o = \begin{bmatrix} \mathbf{u}_{1c} \\ \mathbf{u}_{2c} \end{bmatrix} \quad \dot{\mathbf{u}}_o = \begin{bmatrix} \dot{\mathbf{u}}_{1c} \\ \dot{\mathbf{u}}_{2c} \end{bmatrix} \qquad (19).$$

The velocities of the collision partners, at the terminus of closure, are equal.

Laplace transform and Eigensolution for the undamped equations of motion

The unilateral Laplace transform for a time domain function f(t) that is integrable over the domain $[0, \infty)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad (20).$$

Taking the Laplace transform of equation (11) converts the differential equations of motion in the time domain into algebraic equations in the Laplace domain.

$$\mathbf{M}\left(s^2\mathbf{u}(s) - s\mathbf{u}_o - \dot{\mathbf{u}}_o\right) + \mathbf{K}\mathbf{u}(s) = \frac{\mathbf{F}_{ca}}{s} + \frac{\mathbf{F}_{\Delta ua}}{s} + \mathcal{L}\left(\mathbf{F}_{ext}\right) \qquad (21).$$

Rearranging equation (21):

$$\left(s^2\mathbf{M} + \mathbf{K}\right)\mathbf{u}(s) = s\mathbf{M}\mathbf{u}_o + \mathbf{M}\dot{\mathbf{u}}_o + \frac{\mathbf{F}_{ca}}{s} + \frac{\mathbf{F}_{\Delta ua}}{s} + \mathcal{L}\left(\mathbf{F}_{ext}\right) \qquad (22).$$

The dynamic stiffness matrix $\mathbf{D}(s)$ and the transfer function $\mathbf{H}(s)$ are defined as:

$$\mathbf{D}(s) = \mathbf{H}^{-1}(s) = s^2\mathbf{M} + \mathbf{K} \qquad (23).$$

For a complex valued function with characteristic polynomial of the order m and degrees of freedom N, $m = 2N + p$; $p \geq 0$. For a complex valued non-zero square matrix under the case of having m Eigenvalues appearing in 2N complex conjugate pairs, the Eigenvalues are s_j and $\bar{s}_j$ with corresponding Eigenvectors $\mathbf{z}_j$ and $\bar{\mathbf{z}}_j$ for $\forall j = 1, \ldots, 2N$. This holds for the undamped problem. The Eigenproblem for any such matrix $\mathbf{X}(s)$, $s \in \mathbb{C}$, can be expressed in the following manner where $\nu_k(s)$ is the distinct k^{th} Eigenvalue with corresponding distinct Eigenvector $\phi_k(s)$.

$$\mathbf{X}(s)\phi_k(s) = \nu_k(s)\phi_k(s) \quad \forall k = 1, \ldots, N \qquad (24).$$

Because the Eigenvalues, $\nu_k(s)$, are distinct, the Eigenvectors are orthogonal and the following relationship holds:

$$\phi_j^T(s)\phi_k(s) = \delta_{kj} \quad \forall k, j = 1, \ldots, N \qquad (25).$$

Equation (24) can then be rewritten as:

$$\begin{aligned} \phi_j^T(s)\mathbf{X}(s)\phi_k(s) &= \nu_k(s)\delta_{kj} \\ \mathbf{\Phi}^T(s)\mathbf{X}(s)\mathbf{\Phi}(s) &= \boldsymbol{\nu}(s) \end{aligned} \qquad (26).$$

Equation (26) is the Eigendecomposition of **X**(s). Substitution of the dynamic stiffness matrix **D**(s) for **X**(s) results in the following solutions for **D**(s) and **H**(s).

$$\begin{aligned} \mathbf{D}(s) &= \mathbf{\Phi}(s)\boldsymbol{\nu}(s)\mathbf{\Phi}^T(s) \\ &= \sum_{k=1}^{N}\left(\phi_k(s)\nu_k(s)\phi_k^T(s)\right) \\ \mathbf{H}(s) &= \mathbf{D}^{-1}(s) \\ &= \mathbf{\Phi}(s)\boldsymbol{\nu}^{-1}(s)\mathbf{\Phi}^T(s) \\ &= \sum_{k=1}^{N}\left(\phi_k(s)\nu_k^{-1}(s)\phi_k^T(s)\right) \end{aligned} \qquad (27).$$

The use of the residue theorem allows for the transfer function to be written in the form shown by equation (28),which avoids the necessity for a partial fraction expansion of the transfer function.

$$\mathbf{H}(s) = \sum_{j=1}^{m}\left(\frac{\mathbf{R}_j}{s - s_j}\right) \qquad (28).$$

When $s = s_j$, the rth Eigenvalue, ν_r, is $\nu_r(s_j)$, which equals zero, while all other Eigenvalues are non-zero and the corresponding Eigenvector $\phi_r(s_j) = \mathbf{z}_j$. Adhikari [13] has shown that the residue $\mathbf{R}_j$ and the transfer function **H**(s) may be written as:

$$\mathbf{R}_j = \frac{\mathbf{z}_j\mathbf{z}_j^T}{\mathbf{z}_j^T\dfrac{\partial\mathbf{D}(s_j)}{\partial s_j}\mathbf{z}_j} = \gamma_j\mathbf{z}_j\mathbf{z}_j^T \qquad (29).$$

The transfer function **H**(s) then becomes:

$$\mathbf{H}(s) = \sum_{j=1}^{m}\left(\frac{\gamma_j\mathbf{z}_j\mathbf{z}_j^T}{s - s_j}\right) \qquad (30).$$

Adhikari [13] has also shown that for the case in which there exists 2N eigenvalues in complex conjugate form, among the m Eigenvalues, equation (30) can be written as:

$$\mathbf{H}(s) = \sum_{j=1}^{N}\left(\frac{\gamma_j\mathbf{z}_j\mathbf{z}_j^T}{s - s_j} + \frac{\overline{\gamma}_j\overline{\mathbf{z}}_j\overline{\mathbf{z}}_j^T}{s - \overline{s}_j}\right) + \sum_{j=2N+1}^{m}\left(\frac{\gamma_j\mathbf{z}_j\mathbf{z}_j^T}{s - s_j}\right) \qquad (31).$$

For the undamped case such that the characteristic polynomial of **D**(s) of second order and zero order terms in s, m = 2N and p = 0. The second term to the right of the equality in (31) becomes zero secondary to the limits on the summation operator. The partial derivative of the dynamic stiffness matrix, evaluated at s_j, with respect to s_j becomes:

$$\frac{\partial\mathbf{D}(s)}{\partial s_j} = 2s_j\mathbf{M} \qquad (32).$$

Correspondingly:

$$\gamma_j = \frac{1}{\mathbf{z}_j^T\dfrac{\partial\mathbf{D}(s_j)}{\partial s_j}\mathbf{z}_j} = \frac{1}{2s_j\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j} \qquad (33).$$

The transfer function then becomes:

$$\mathbf{H}(s) = \frac{1}{2}\sum_{j=1}^{N}\left(\frac{\mathbf{z}_j\mathbf{z}_j^T}{s_j\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j(s - s_j)} + \frac{\overline{\mathbf{z}}_j\overline{\mathbf{z}}_j^T}{\overline{s}_j\overline{\mathbf{z}}_j^T\mathbf{M}\overline{\mathbf{z}}_j(s - \overline{s}_j)}\right) \qquad (34).$$

For the undamped system, the characteristic polynomial of **D**(s) results in a solution for the Eigenvalue problem where $s_j = a_j + jb_j = jb_j$, where $b_j \in \mathbb{R}, s_j \in \mathbb{C}$ for $\forall j$. As a result:

$$s_j = jb_j \quad \overline{s}_j = -jb_j = -s_j \qquad (35).$$

The Eigenvalues of **D**(s) are determined by taking the determinant of **D**(s) and solving for the values of s_j such that det[**D**(s)] is equal to zero.

$$\det\left[\mathbf{D}(s)\right] = \det\left[s^2\mathbf{M} + \mathbf{K}\right] = 0 \qquad (36).$$

Substitution of the terms from **M** and **K** into equation (36) results in the following form of the characteristic polynomial.

$$s^2\left(m_1m_2s^2 + k_{eff}(m_1 + m_2)\right) = 0 \qquad (37).$$

The Eigenvalues are the roots of equation (37)

$$s_1 = 0 \quad s_2 = \pm j\sqrt{\left(\frac{m_1 + m_2}{m_1 m_2}\right)k_{eff}} \qquad (38).$$

Correspondingly:

$$b_1 = 0 \quad b_2 = \sqrt{\left(\frac{m_1 + m_2}{m_1 m_2}\right)k_{eff}} \qquad (39).$$

The Eigenvector problem for **D**(s) is given as:

$$\mathbf{D}\left(s_j\right)\mathbf{z}_j = 0 \qquad (40).$$

Substitution for the positions of $\mathbf{D}(s_j)$ in terms of the components of **M** and the positions of **K** results in the following:

$$\begin{bmatrix} s_j^2 m_1 + k_{eff} & -k_{eff} \\ -k_{eff} & s_j^2 m_2 + k_{eff} \end{bmatrix}\begin{bmatrix} z_{j1} \\ z_{j2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (41).$$

Normalizing the solution such that z_{j2} is unity:

$$\mathbf{z}_j = \begin{bmatrix} z_{j1} \\ z_{j2} \end{bmatrix} = \begin{bmatrix} \dfrac{k_{eff}}{\left(s_j^2 m_1 + k_{eff}\right)} \\ 1 \end{bmatrix} \qquad (42).$$

For the conjugate of s_j:

$$\overline{\mathbf{z}}_j = \begin{bmatrix} \overline{z}_{j1} \\ \overline{z}_{j2} \end{bmatrix} = \begin{bmatrix} \dfrac{k_{eff}}{\overline{s}_j^2 m_1 + k_{eff}} \\ 1 \end{bmatrix} \qquad (43).$$

From equation (35), it can be seen that the square of the Eigenvalue $s_j = jb_j$ and its complex conjugate are equal. Therefore, the Eigenvector $\mathbf{z}_j$ and its complex conjugate are equal for the undamped case.

$$\mathbf{z}_j = \begin{bmatrix} z_{j1} \\ z_{j2} \end{bmatrix} = \begin{bmatrix} \dfrac{k_{eff}}{\left(k_{eff} - b_j^2 m_1\right)} \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{z}_{j1} \\ \overline{z}_{j2} \end{bmatrix} = \overline{\mathbf{z}}_j \qquad (44).$$

The Eigenvectors corresponding to s_1 and s_2 thus become:

$$\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{z}_2 = \begin{bmatrix} -m_1^{-1} m_2 \\ 1 \end{bmatrix} \qquad (45).$$

Equation (34) requires the evaluation of the scalars $\mathbf{z}_j^T \mathbf{M} \mathbf{z}_j$ and $\overline{\mathbf{z}}_j^T \mathbf{M} \overline{\mathbf{z}}_j$.

$$\begin{aligned} \mathbf{z}_j^T \mathbf{M} \mathbf{z}_j &= \begin{bmatrix} z_{j1} & z_{j2} \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} z_{j1} \\ z_{j2} \end{bmatrix} \\ &= \begin{bmatrix} m_1 z_{j1} & m_2 z_{j2} \end{bmatrix}\begin{bmatrix} z_{j1} \\ z_{j2} \end{bmatrix} \\ &= m_1 z_{j1}^2 + m_2 z_{j2}^2 \end{aligned} \qquad (46).$$

$$\begin{aligned} \overline{\mathbf{z}}_j^T \mathbf{M} \overline{\mathbf{z}}_j &= \begin{bmatrix} \overline{z}_{j1} & \overline{z}_{j2} \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \overline{z}_{j1} \\ \overline{z}_{j2} \end{bmatrix} \\ &= \begin{bmatrix} m_1 \overline{z}_{j1} & m_2 \end{bmatrix}\begin{bmatrix} \overline{z}_{j1} \\ \overline{z}_{j2} \end{bmatrix} \\ &= m_1 \overline{z}_{j1}^2 + m_2 \overline{z}_{j2} \\ &= m_1 z_{j1}^2 + m_2 z_{j2}^2 \end{aligned} \qquad (47).$$

Terms of the form $\mathbf{z}_j \mathbf{z}_j^T$ and $\overline{\mathbf{z}}_j \overline{\mathbf{z}}_j^T$ are matrices:

$$\begin{aligned} \mathbf{z}_j \mathbf{z}_j^T &= \overline{\mathbf{z}}_j \overline{\mathbf{z}}_j^T \\ &= \begin{bmatrix} z_{j1} \\ z_{j1} \end{bmatrix}\begin{bmatrix} z_{j1} & z_{j1} \end{bmatrix} = \begin{bmatrix} z_{j1}^2 & z_{j1} z_{j2} \\ z_{j1} z_{j2} & z_{j2}^2 \end{bmatrix} \end{aligned} \qquad (48).$$

Equation (34) can then be rewritten as:

$$\begin{aligned} \mathbf{H}(s) &= \sum_{j=1}^{N}\left(\frac{\begin{bmatrix} z_{j1}^2 & z_{j1} z_{j2} \\ z_{j1} z_{j2} & z_{j2}^2 \end{bmatrix}}{m_1 z_{j1}^2 + m_2 z_{j2}^2}\left(\frac{1}{s^2 + b_j^2}\right)\right) \\ &= \sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{1}{s^2 + b_j^2}\right)\right) \end{aligned} \qquad (49).$$

From equation (45), the values of $\mathbf{c}_1$ and $\mathbf{c}_2$ are:

$$\mathbf{c}_1 = \frac{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{m_1 + m_2} \quad \mathbf{c}_2 = \frac{\begin{bmatrix} m_1^{-2} m_2^2 & -m_1^{-1} m_2 \\ -m_1^{-1} m_2 & 1 \end{bmatrix}}{m_1^{-1} m_2^2 + m_2} \qquad (50).$$

Substitution of equation (49) into equation (22) yields the solution for the displacement of the collision partners in the Laplace domain.

$$\mathbf{u}(s)=\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{s}{s^2+b_j^2}\right)\right)\right]\mathbf{M}\mathbf{u_o}+\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{1}{s^2+b_j^2}\right)\right)\right]\mathbf{M}\dot{\mathbf{u}}_\mathbf{o}+\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{1}{s\left(s^2+b_j^2\right)}\right)\right)\right]\left(\mathbf{F}_{ca}+\mathbf{F}_{\Delta ua}\right)+\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\} \quad (51).$$

The inverse Laplace transform of equation (51) yields the time domain displacement solution. The first and second derivatives, with respect to time, yield the velocity and acceleration solution, respectively, in the time domain.

$$\mathbf{u}(t)=\sum_{j=1}^{N}\left(\mathbf{c}_j\mathbf{M}\mathbf{u_o}\cos\left(b_jt\right)\right)+\sum_{j=1}^{N}\left(\mathbf{c}_j\mathbf{M}\dot{\mathbf{u}}_\mathbf{o}\frac{\sin\left(b_jt\right)}{b_j}\right)+\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\mathbf{F}_{ca}+\mathbf{F}_{\Delta ua}\right)\left(\frac{1-\cos\left(b_jt\right)}{b_j^2}\right)\right)+\mathcal{L}^{-1}\left\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\right\} \quad (52).$$

$$\dot{\mathbf{u}}(t)=-\sum_{j=1}^{N}\left(\mathbf{c}_j\mathbf{M}\mathbf{u_o}b_j\sin\left(b_jt\right)\right)+\sum_{j=1}^{N}\left(\mathbf{c}_j\mathbf{M}\dot{\mathbf{u}}_\mathbf{o}\cos\left(b_jt\right)\right)+\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\mathbf{F}_{ca}+\mathbf{F}_{\Delta ua}\right)\left(\frac{\sin\left(b_jt\right)}{b_j}\right)\right)+\frac{d}{dt}\left(\mathcal{L}^{-1}\left\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\right\}\right) \quad (53).$$

$$\ddot{\mathbf{u}}(t)=-\sum_{j=1}^{N}\left(\mathbf{c}_j\mathbf{M}\mathbf{u_o}b_j^2\cos\left(b_jt\right)\right)-\sum_{j=1}^{N}\left(\mathbf{c}_j\mathbf{M}\dot{\mathbf{u}}_\mathbf{o}b_j\sin\left(b_jt\right)\right)+\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\mathbf{F}_{ca}+\mathbf{F}_{\Delta ua}\right)\cos\left(b_jt\right)\right)+\frac{d^2}{dt^2}\left(\mathcal{L}^{-1}\left\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\right\}\right) \quad (54).$$

For the case in which a rigid body mode of motion is indication (i.e. $s_j = 0$), the terms associated with j = 1 can be separated from the summation operator in equation (51) and evaluated separately.

Specific issues regarding closure

Using the initial conditions, described previously, for a typical closure scenario, equation (51) reduces to the following form:

$$\mathbf{u}(s)=\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{1}{s^2+b_j^2}\right)\right)\right]\mathbf{M}\dot{\mathbf{u}}_\mathbf{o}+\mathbf{H}(s)\mathcal{L}\left(\mathbf{F}_{ext}\right) \quad (55).$$

Applying the summation operator and substituting for the constituent terms results in the following form of the displacement solution in the Laplace domain.

$$\mathbf{u}(s)=\frac{1}{m_1+m_2}\cdot\left(\begin{bmatrix}m_1\dot{u}_{1o}+m_2\dot{u}_{2o}\\ m_1\dot{u}_{1o}+m_2\dot{u}_{2o}\end{bmatrix}\left(\frac{1}{s^2}\right)+\begin{bmatrix}m_2\left(\dot{u}_{1o}-\dot{u}_{2o}\right)\\ m_1\left(\dot{u}_{2o}-\dot{u}_{1o}\right)\end{bmatrix}\left(\frac{1}{s^2+b_2^2}\right)\right)+\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\} \quad (56).$$

The time domain solutions are obtained as before.

$$\mathbf{u}(t)=\frac{1}{m_1+m_2}\cdot\left(\begin{bmatrix}m_1\dot{u}_{1o}+m_2\dot{u}_{2o}\\ m_1\dot{u}_{1o}+m_2\dot{u}_{2o}\end{bmatrix}t+\begin{bmatrix}m_2\left(\dot{u}_{1o}-\dot{u}_{2o}\right)\\ m_1\left(\dot{u}_{2o}-\dot{u}_{1o}\right)\end{bmatrix}\left(\frac{\sin\left(b_2t\right)}{b_2}\right)\right)+\mathcal{L}^{-1}\left\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\right\} \quad (57).$$

$$\dot{\mathbf{u}}(t)=\frac{1}{m_1+m_2}\cdot\left(\begin{bmatrix}m_1\dot{u}_{1o}+m_2\dot{u}_{2o}\\ m_1\dot{u}_{1o}+m_2\dot{u}_{2o}\end{bmatrix}+\begin{bmatrix}m_2\left(\dot{u}_{1o}-\dot{u}_{2o}\right)\\ m_1\left(\dot{u}_{2o}-\dot{u}_{1o}\right)\end{bmatrix}\cos\left(b_2t\right)\right)+\frac{d}{dt}\left(\mathcal{L}^{-1}\left\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\right\}\right) \quad (58).$$

$$\ddot{\mathbf{u}}(t) = -\frac{b_2}{m_1+m_2}\begin{bmatrix} m_2(\dot{u}_{1o}-\dot{u}_{2o}) \\ m_1(\dot{u}_{2o}-\dot{u}_{1o}) \end{bmatrix}\sin(b_2 t) + \frac{d^2}{dt^2}\left(\mathcal{L}^{-1}\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F_{ext}}\}\}\right) \quad (59).$$

In the case in which there are no externally applied unbalanced forces on either collision partner, the time at which closure terminates can readily be found as $t_c = \pi / (2b_2)$. An analytical closed form solution for t_c can also be found for cases in which $\mathbf{F_{ext}}$ is not zero and for which the time domain velocity solution includes other functions of time besides $\cos(b_2 t)$. The solution in such cases is determined by taking the Laplace transform of equation (58), setting the velocities of the collision partners equal to each other and solving for s. The inverse Laplace transform of s yields the solution $t_c\,\boldsymbol{\delta}(t_c)$ where $\boldsymbol{\delta}$ is the Dirac Delta function and $\boldsymbol{\delta}(t_c)$ has a value of unity at t_c and zero elsewhere. With the value of t_c known, the displacements at the terminus of closure can be solved for using equation (57) and the velocities at the terminus of closure can be solved for using equation (58) The displacement of the common collision interface at the terminus of closure can then be solved for using equation (7) The peak deflections can then be determined from equation (6) and the magnitude of the peak collision force can be determined from equation (5)

Specific issues regarding separation

As noted previously, the terminus conditions regarding closure represent the initial conditions for separation. Furthermore, the equations defined in (17) become operative. Because the time shift of $t^* = t - t_c$ is being used, the conjoined loading and unloading solutions, require the substitution of the latter for the former for a complete solution in terms of t. The tilde notation is employed to distinguish parameters associated with the unload (separation) phase from those associated with the load (closure) phase. We sequentially expand the three terms to the right of the equality of equation (51)

$$\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{s}{s^2+\tilde{b}_j^{\,2}}\right)\right)\right]\mathbf{M}\mathbf{u}_\mathbf{c} = \frac{1}{m_1+m_2}\left(\begin{bmatrix} m_1 u_{1c}+m_2 u_{2c} \\ m_1 u_{1c}+m_2 u_{2c} \end{bmatrix}\left(\frac{1}{s}\right) + \begin{bmatrix} m_2(u_{1c}-u_{2c}) \\ m_1(u_{2c}-u_{1c}) \end{bmatrix}\left(\frac{s}{s^2+\tilde{b}_2^{\,2}}\right)\right) \quad (60).$$

$$\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{1}{s^2+\tilde{b}_j^{\,2}}\right)\right)\right]\mathbf{M}\dot{\mathbf{u}}_\mathbf{c} = \frac{1}{m_1+m_2}\left(\begin{bmatrix} m_1 \dot{u}_{1c}+m_2 \dot{u}_{2c} \\ m_1 \dot{u}_{1c}+m_2 \dot{u}_{2c} \end{bmatrix}\left(\frac{1}{s^2}\right) + \begin{bmatrix} m_2(\dot{u}_{1c}-\dot{u}_{2c}) \\ m_1(\dot{u}_{2c}-\dot{u}_{1c}) \end{bmatrix}\left(\frac{1}{s^2+\tilde{b}_2^{\,2}}\right)\right) \quad (61).$$

$$\left[\sum_{j=1}^{N}\left(\mathbf{c}_j\left(\frac{1}{s\left(s^2+b_j^{\,2}\right)}\right)\right)\right]\left(\mathbf{F_{ca}}+\mathbf{F_{\Delta ua}}\right) = \frac{\left(F_{cp}+k_{eff}(u_{2c}-u_{1c})\right)}{m_1+m_2}\cdot\left(\begin{bmatrix} -1+1 \\ -1+1 \end{bmatrix}\left(\frac{1}{s^3}\right) + \begin{bmatrix} -m_1^{-1}m_2-1 \\ m_1 m_2^{-1}+1 \end{bmatrix}\left(\frac{1}{s\left(s^2+\tilde{b}_2^{\,2}\right)}\right)\right) \quad (62).$$

In equation (61), the second term within the parenthesis on the right of the equality is zero as the differences in the velocities of the collision partners at closure is zero. In equation (62), the first term within the parenthesis on the right of the equality is clearly zero (i.e. -1+1 = 0). Taking the inverse Laplace transform of equations (60) through (62) and summing the resultants (along with the general form for the externally applied unbalanced forces) gives the specific form of the displacement solution in the time domain.

$$\mathbf{u}\left(t^*\right)=\frac{1}{m_1+m_2}\cdot\left(\begin{array}{l}\begin{bmatrix}m_1u_{1c}+m_2u_{2c}\\ m_1u_{1c}+m_2u_{2c}\end{bmatrix}+\\ \begin{bmatrix}m_2\left(u_{1c}-u_{2c}\right)\\ m_1\left(u_{2c}-u_{1c}\right)\end{bmatrix}\cos\left(\tilde{b}_2t^*\right)+\\ \begin{bmatrix}m_1\dot{u}_{1c}+m_2\dot{u}_{2c}\\ m_1\dot{u}_{1c}+m_2\dot{u}_{2c}\end{bmatrix}t^*+\\ \begin{bmatrix}-m_1^{-1}m_2-1\\ m_1m_2^{-1}+1\end{bmatrix}\cdot\\ \left(\dfrac{1-\cos\left(\tilde{b}_2t^*\right)}{\tilde{b}_2^{\,2}}\right)\cdot\\ \begin{pmatrix}F_{cp}+\\ k_{eff}\left(u_{2c}-u_{1c}\right)\end{pmatrix}\end{array}\right)+\mathcal{L}^{-1}\left(\mathbf{H}(s)\mathcal{L}\left(\mathbf{F_{ext}}\right)\right) \quad (63).$$

The solutions for the velocity and acceleration are:

$$\dot{\mathbf{u}}\left(t^*\right)=\frac{1}{m_1+m_2}\left(\begin{array}{l}-\begin{bmatrix}m_2\left(u_{1c}-u_{2c}\right)\\ m_1\left(u_{2c}-u_{1c}\right)\end{bmatrix}\cdot\\ \tilde{b}_2\sin\left(\tilde{b}_2t^*\right)+\\ \begin{bmatrix}m_1\dot{u}_{1c}+m_2\dot{u}_{2c}\\ m_1\dot{u}_{1c}+m_2\dot{u}_{2c}\end{bmatrix}+\\ \begin{bmatrix}-m_1^{-1}m_2-1\\ m_1m_2^{-1}+1\end{bmatrix}\cdot\\ \left(\dfrac{\sin\left(\tilde{b}_2t^*\right)}{\tilde{b}_2}\right)\cdot\\ \left(F_{cp}+k_{eff}\left(u_{2c}-u_{1c}\right)\right)\end{array}\right)+\frac{d}{dt}\mathcal{L}^{-1}\left(\mathbf{H}(s)\mathcal{L}\left(\mathbf{F_{ext}}\right)\right) \quad (64).$$

$$\ddot{\mathbf{u}}\left(t^*\right)=\frac{1}{m_1+m_2}\left(\begin{array}{l}-\begin{bmatrix}m_2\left(u_{1c}-u_{2c}\right)\\ m_1\left(u_{2c}-u_{1c}\right)\end{bmatrix}\cdot\\ \tilde{b}_2^{\,2}\cos\left(\tilde{b}_2t^*\right)+\\ \begin{bmatrix}-m_1^{-1}m_2-1\\ m_1m_2^{-1}+1\end{bmatrix}\cos\left(\tilde{b}_2t^*\right)\cdot\\ \left(F_{cp}+k_{eff}\left(u_{2c}-u_{1c}\right)\right)\end{array}\right)+\frac{d^2}{dt^2}\mathcal{L}^{-1}\left(\mathbf{H}(s)\mathcal{L}\left(\mathbf{F_{ext}}\right)\right) \quad (65).$$

The time at which separation occurs is the time at which the accelerations of the collision partners due to the collision force alone, drops to zero. By the form of equation (65), this occurs at $t_s^* = \pi / (2b_2)$.

ANALYSIS

We first consider the case in which no net externally applied forces are present to develop a reference. We define the constant terms $\mathbf{A}_m \forall m$, which are readily defined in the following development. From equation (39), $b_{11} = b_{12} = 0$ while b_{21} and b_{22}, where the second subscript represents loading and unloading, respectively, are non-zero.

During closure, from the form of equation (57), with the term associated with the external forces set to zero, the displacement for each collision partner may be written as

$$\mathbf{u}_j(t)=\mathbf{A}_o t+\mathbf{A}_{1j}\sin\left(b_{21}t\right) \quad (66).$$

The velocity and acceleration during closure are:

$$\begin{aligned}\dot{\mathbf{u}}_j(t)&=\mathbf{A}_o+\mathbf{A}_{1j}b_{21}\cos\left(b_{21}t\right)\\ \ddot{\mathbf{u}}_j(t)&=-\mathbf{A}_{1j}b_{21}^{\,2}\sin\left(b_{21}t\right)\end{aligned} \quad (67).$$

The displacement of the common collision interface during closure is (where the stiffness values are the loading stiffnesses):

$$\mathbf{u}_{IV}(t)=\mathbf{A}_o t+\frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j}\sin\left(b_{21}t\right) \quad (68).$$

The structural deflection experienced by each collision partner during closure is:

$$\begin{aligned}\boldsymbol{\delta}_1(t)&=\left(\mathbf{A}_{11}-\frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j}\right)\sin\left(b_{21}t\right)\\ \boldsymbol{\delta}_2(t)&=\left(\frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j}-\mathbf{A}_{12}\right)\sin\left(b_{21}t\right)\end{aligned} \quad (69).$$

As noted before, the terminus of closure is at $t = t_c = 0.5\pi b_{21}^{-1}$. The common velocity reached at the terminus of closure, $\mathbf{v}_{com}$ is equal to $\mathbf{A}_o$ and the peak acceleration for each collision partner is $-\mathbf{A}_{1j}b_{21}^{\,2}$.

The peak displacements for each collision partner and the common collision interface are:

$$\mathbf{u}_j\left(t_c\right) = \mathbf{u}_{jc} = 0.5\mathbf{A}_o\pi b_{21}^{-1} + \mathbf{A}_{1j}$$
$$\mathbf{u}_{IV}\left(t_c\right) = \mathbf{u}_{IVc} = 0.5\mathbf{A}_o\pi b_{21}^{-1} + \frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j} \quad (70).$$

The peak deflections are:

$$\boldsymbol{\delta}_1\left(t_c\right) = \boldsymbol{\delta}_{1d} = \left(\mathbf{A}_{11} - \frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j}\right)$$
$$\boldsymbol{\delta}_2\left(t_c\right) = \boldsymbol{\delta}_{2d} = \left(\frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j} - \mathbf{A}_{12}\right) \quad (71).$$

Finally, the magnitude of the peak collision force is:

$$\left|\mathbf{F}_{cp}\right| = k_1\left(\mathbf{A}_{11} - \frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j}\right)$$
$$= k_2\left(\frac{\sum_j k_j\mathbf{A}_{1j}}{\sum_j k_j} - \mathbf{A}_{12}\right) \quad (72).$$

During separation, $t^* = t - t_c$. From equation (63) it is evident that the form of the displacement for each collision partner during unloading is:

$$\mathbf{u}_j\left(t^*\right) = \mathbf{u}_{jc} + \mathbf{A}_o t^* + \mathbf{A}_{2j}\cos\left(b_{22}t^*\right) \quad (73).$$

The solutions for the velocity and acceleration are:

$$\dot{\mathbf{u}}_j\left(t^*\right) = \mathbf{A}_o - \mathbf{A}_{2j}b_{22}\sin\left(b_{22}t^*\right) \quad (74).$$

$$\ddot{\mathbf{u}}_j\left(t^*\right) = -\mathbf{A}_{2j}b_{22}^2\cos\left(b_{22}t^*\right) \quad (75).$$

The displacement of the common collision interface during separation is (the stiffness values are the unloading stiffnesses):

$$\mathbf{u}_{IV}\left(t^*\right) = \mathbf{A}_o t^* + \frac{\sum_j k_j\left(u_{jc} + A_{2j}\cos\left(b_{22}t^*\right)\right)}{\sum_j k_j} \quad (76)$$

The deflection experienced by each collision partner during separation is:

$$\boldsymbol{\delta}_1\left(t^*\right) = \mathbf{u}_{1c} + \mathbf{A}_{21}\cos\left(b_{22}t^*\right) - \frac{\sum_j k_j\left(u_{jc} + A_{2j}\cos\left(b_{22}t^*\right)\right)}{\sum_j k_j} \quad (77).$$

$$\boldsymbol{\delta}_2\left(t^*\right) = \frac{\sum_j k_j\left(u_{jc} + A_{2j}\cos\left(b_{22}t^*\right)\right)}{\sum_j k_j} - \mathbf{u}_{2c} - \mathbf{A}_{22}\cos\left(b_{22}t^*\right) \quad (78).$$

The terminus of separation, as noted previously, occurs when the collision force drops to zero magnitude. This occurs at $t^* = 0.5\pi b_{22}^{-1}$. The displacements at the terminus of separation are:

$$\mathbf{u}_j\left(t_s^*\right) = \mathbf{u}_{jc} + 0.5\mathbf{A}_o\pi b_{22}^{-1} \quad (79).$$

$$\mathbf{u}_{IV}\left(t_s^*\right) = 0.5\mathbf{A}_o\pi b_{22}^{-1} + \frac{\sum_j k_j u_{jc}}{\sum_j k_j} \quad (80).$$

The residual damage depths are:

$$\boldsymbol{\delta}_1\left(t_s^*\right) = \mathbf{u}_{1c} - \frac{\sum_j k_j u_{jc}}{\sum_j k_j} \quad (81).$$

$$\boldsymbol{\delta}_2\left(t_s^*\right) = \frac{\sum_j k_j u_{jc}}{\sum_j k_j} - \mathbf{u}_{jc} \quad (82).$$

Linear net external forces

We define the net externally applied force on the striking vehicle as $\mathbf{F}_{ext1}(t) = \mathbf{g}_{o1} + \mathbf{g}_{11}t$. When this force is time invariant (i.e. $\mathbf{g}_{11} = \mathbf{0}$) then the model reduces to a simple constant force model (i.e. $\mathbf{g}_{01} = -f\mu m_1 g$ for constant braking). The external force vector, in the Laplace domain, in equation (51), becomes:

$$\mathcal{L}\{\mathbf{F}_{ext}\} = \begin{pmatrix} \frac{\mathbf{g}_{o1}}{s} + \frac{\mathbf{g}_{11}}{s^2} \\ \frac{\mathbf{g}_{o2}}{s} + \frac{\mathbf{g}_{12}}{s^2} \end{pmatrix} \quad (83).$$

Multiplying the external force vector, in the Laplace domain, by the transfer function and taking the inverse Laplace transform results in the following time domain solution (using the subscript notation as before):

$$\mathcal{L}^{-1}\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\} = \left(6b_2^3 m_j (m_1 + m_2)\right)^{-1} \cdot \begin{pmatrix} b_2 \begin{pmatrix} 6(g_{oj}m_k - g_{ok}m_j) + \\ 6t(g_{1j}m_k - g_{1k}m_j) + \\ b_2^2 m_j t^3 (g_{11} + g_{12}) + \\ 3b_2^2 m_j t^2 (g_{o2} + g_{o1}) \end{pmatrix} + \\ 6b_2 (g_{ok}m_j - g_{oj}m_k)\cos(b_2 t) + \\ 6(g_{1k}m_j - g_{1j}m_k)\sin(b_2 t) \end{pmatrix} \quad (84).$$

As expected, substitution of t = 0 into equation (84) results in zero displacement due to the externally applied loads. The corresponding first and second time derivatives, required for the velocity and acceleration solution, respectively, are:

$$\frac{d}{dt}\mathcal{L}^{-1}\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\} = \left(6b_2^3 m_j (m_1 + m_2)\right)^{-1} \cdot \begin{pmatrix} b_2 \begin{pmatrix} 6(g_{1j}m_k - g_{1k}m_j) + \\ 3b_2^2 m_j t^2 (g_{11} + g_{12}) + \\ 6b_2^2 m_j t (g_{o2} + g_{o1}) \end{pmatrix} - \\ 6b_2^2 (g_{ok}m_j - g_{oj}m_k)\sin(b_2 t) + \\ 6b_2 (g_{1k}m_j - g_{1j}m_k)\cos(b_2 t) \end{pmatrix} \quad (85).$$

$$\frac{d^2}{dt^2}\mathcal{L}^{-1}\{\mathbf{H}(s)\mathcal{L}\{\mathbf{F}_{ext}\}\} = \left(6b_2^3 m_j (m_1 + m_2)\right)^{-1} \cdot \begin{pmatrix} b_2 \begin{pmatrix} 6b_2^2 m_j t (g_{11} + g_{12}) + \\ 6b_2^2 m_j (g_{o2} + g_{o1}) \end{pmatrix} - \\ 6b_2^3 (g_{ok}m_j - g_{oj}m_k)\cos(b_2 t) - \\ 6b_2^2 (g_{1k}m_j - g_{1j}m_k)\sin(b_2 t) \end{pmatrix} \quad (86).$$

The time at which closure occurs can be readily obtained by setting the velocities of the collision partners equal to each other and solving for $t = t_c$. For zero initial displacement conditions the solution is determinate in closed form but is of such length as to preclude its inclusion herein.

The solutions for the displacement and acceleration of each collision partner at the terminus of closure are obtained, directly, by substitution of the time at which closure terminates into equations (84) and (86) respectively. Using the terminus of closure conditions as the initial conditions for the separation phase, the time domain displacement solution is:

$$\mathbf{u}_j(t^*) = \frac{1}{6}(m_1 + m_2)^{-1} \cdot \left(c_1 + \frac{1}{(\tilde{b}_2^2 m_j)c_2}\right) \quad (87).$$

Where:

$$c_1 = 6\begin{pmatrix} (\dot{\mathbf{u}}_{1c}m_1 + \dot{\mathbf{u}}_{2c}m_2)(t^* + 1) + \\ (-1)^j \begin{pmatrix} \mathbf{F}_{cp} + \\ \tilde{k}_{eff}(\mathbf{u}_{2c} - \mathbf{u}_{1c}) \end{pmatrix} \cdot \\ (\cos(b_2 t^*) - 1) + \\ m_j (\mathbf{u}_{jc} - \mathbf{u}_{kc})\cos(b_2 t^*) \end{pmatrix} \quad (88).$$

$$c_2 = b_2 \begin{pmatrix} 6(\mathbf{g}_{oj}m_k - \mathbf{g}_{ok}m_j) \\ +6(\mathbf{g}_{1j}m_k - \mathbf{g}_{1k}m_j)t^* \\ +3b_2^2(\mathbf{g}_{o1}m_j + \mathbf{g}_{o2}m_j)t^{*2} + \\ b_2^2(\mathbf{g}_{11}m_j + \mathbf{g}_{12}m_j)t^{*3} \end{pmatrix} + (-1)^k 6b_2(\mathbf{g}_{o2}m_1 - \mathbf{g}_{o1}m_2)\cos(b_2 t^*) + 6(\mathbf{g}_{1k}m_j - \mathbf{g}_{1j}m_k)\sin(b_2 t^*) \quad (89).$$

The velocity of each collision partner during separation is:

$$\dot{\mathbf{u}}_j\left(t^*\right)=\frac{1}{6}\left(m_1+m_2\right)^{-1}\cdot\left(\frac{3c_3}{\left(b_2{}^2m_j\right)}+6c_4\right) \quad (90).$$

Where:

$$\begin{aligned} c_3 = {} & 2\left(\mathbf{g}_{1j}m_k-\mathbf{g}_{jk}m_j\right)+2b_2{}^2m_j\left(\mathbf{g}_{o1}+\mathbf{g}_{o2}\right)t^*+ \\ & b_2{}^2m_j\left(\mathbf{g}_{11}+\mathbf{g}_{12}\right)t^{*2}+ \\ & 2\left(g_{1k}m_j-g_{1j}m_k\right)\cos\left(b_2t^*\right)+ \\ & 2b_2\left(\mathbf{g}_{oj}m_k-\mathbf{g}_{ok}m_j\right)\sin\left(b_2t^*\right)+ \end{aligned} \quad (91).$$

$$c_4=\begin{pmatrix} \dot{\mathbf{u}}_{1c}m_1+\dot{\mathbf{u}}_{2c}m_2+ \\ (-1)^j\,\tilde{b}_2m_k\left(\mathbf{u}_{1c}-\mathbf{u}_{2c}\right)\sin\left(\tilde{b}_2t^*\right)+ \\ (-1)^j\left(\dfrac{m_1+m_2}{\tilde{b}_2m_j}\right)\cdot \\ \left(\mathbf{F}_{cp}+\tilde{k}_{eff}\left(\mathbf{u}_{2c}-\mathbf{u}_{1c}\right)\right)\sin\left(\tilde{b}_2t^*\right) \end{pmatrix} \quad (92).$$

Finally, the acceleration of each collision partner during separation is:

$$\ddot{\mathbf{u}}_j\left(t^*\right)=b_2{}^{-1}m_k{}^{-1}\left(m_1+m_2\right)^{-1}\cdot\left(c_5+c_6\right) \quad (93).$$

Where:

$$c_5=b_2\begin{pmatrix} \mathbf{g}_{o1}m_j+\mathbf{g}_{o2}m_k+m_j\left(\mathbf{g}_{11}+\mathbf{g}_{12}\right)m_jt+ \\ \left(\mathbf{g}_{o1}m_k-\mathbf{g}_{o2}m_j\right)\cos\left(b_2t\right) \end{pmatrix} \quad (94).$$

$$\begin{aligned} c_6 = {} & (-1)^j\,b_2\cdot \\ & \begin{pmatrix} \mathbf{F}_{cp}\left(m_1+m_2\right)- \\ \begin{pmatrix} -\tilde{b}_2{}^2m_1m_2+ \\ \tilde{k}_{eff}\left(m_1+m_2\right) \end{pmatrix}\left(\mathbf{u}_{1c}-\mathbf{u}_{2c}\right) \end{pmatrix}\cos\left(\tilde{b}_2t\right)+ \\ & \left(\mathbf{g}_{1j}m_k-\mathbf{g}_{1k}m_j\right)\sin\left(b_2t\right) \end{aligned} \quad (95).$$

The time at which separation terminates can be determined in closed form as before. The solution, however, is exceedingly lengthy and precluded from inclusion due to space limitations. Substitution of the solution for the time at which separation terminates into equation (87) yields the solution for the displacement at the terminus of closure. Similarly, substitution into equation (90) yields the velocity of each collision partner at separation.

EXAMPLE

As a hypothetical example, we consider the inline collision between two collision partners with the following parameters: m_1 = 1500 kg, m_2 = 1900 kg, k_1 (loading phase) = $5 \cdot 10^5$ Nm^{-1}, k_2 (loading phase) = $1.9 \cdot 10^5$ Nm^{-1}, k_1 (unloading phase) = $3 \cdot 10^6$ Nm^{-1}, k_2 (unloading phase) = $5 \cdot 10^6$ Nm^{-1}, zero magnitude initial displacement and with initial velocities of 15 and 4 $msec^{-1}$. The circular frequency for the load phase, using equation (39), is 12.8 sec^{-1}. The load phase displacement response, under no net external forces, for each collision partner and the common collision interface, respectively, are:

$$\begin{aligned} \mathbf{u}_1\left(t\right) &= 8.85t+0.48\sin\left(12.8t\right) \\ \mathbf{u}_2\left(t\right) &= 8.85t-0.38\sin\left(12.8t\right) \\ \mathbf{u}_{IV}\left(t\right) &= 8.85t+0.24\sin\left(12.8t\right) \end{aligned} \quad (96).$$

The load phase velocity and acceleration is obtained by the simple first and second time derivatives of the equations shown in (96). The structural deflection, during the load phase, for each collision partner, is:

$$\begin{aligned} \boldsymbol{\delta}_1\left(t\right) &= 0.24\sin\left(12.8t\right) \\ \boldsymbol{\delta}_2\left(t\right) &= 0.62\sin\left(12.8t\right) \end{aligned} \quad (97).$$

The time at which closure occurs is 0.123 sec with peak displacements of 1.56 m and 0.71 m for each collision partner and with peak deflections of 0.24 m and 0.62 m, respectively. The common velocity at the terminus of closure is 8.85 $msec^{-1}$ and the peak collision force is $1.18 \cdot 10^6$ N. For the unload phase, the circular frequency is 47.3 sec^{-1}. The displacement for each collision partner and the common collision interface, respectively, are:

$$\begin{aligned} \mathbf{u}_1\left(t^*\right) &= 1.53+8.85t^*+0.035\cos\left(47.3t^*\right) \\ \mathbf{u}_2\left(t^*\right) &= 0.73+8.85t^*-0.028\cos\left(47.3t^*\right) \\ \mathbf{u}_{IV}\left(t^*\right) &= 1.33+8.85t^*-0.0042\cos\left(47.3t^*\right) \end{aligned} \quad (98).$$

Separation terminates at t^* = 0.033 sec (t_s = 0.156 sec) with terminus displacement values of 1.83 m and 1.03 m, respectively, and peak deflections of 0.20 m and 0.59 m, respectively. The separation velocity of each collision partner is 7.19 and 10.2 $msec^{-1}$, respectively. Figure 3 depicts the kinematic response for both collision partners.

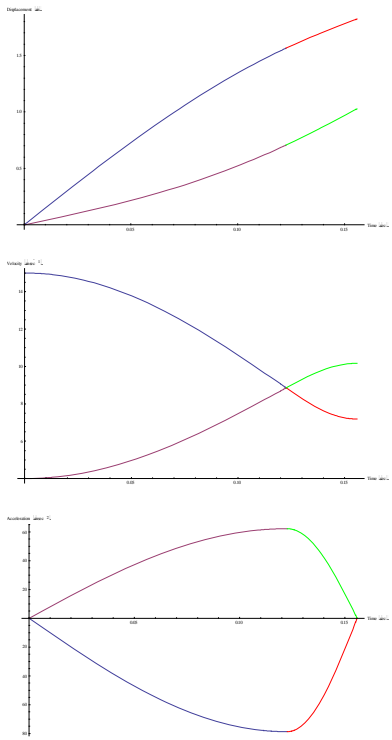

Figure 3. Displacement, velocity and acceleration time histories for the subject example.

Figure 4 depicts the parametric in time force-deflection response for both collision partners.

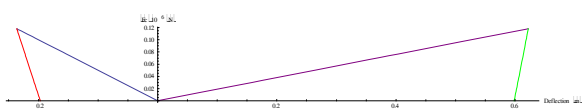

Figure 4. Collision force magnitude as a function of structural deflection for each collision partner.

We consider this problem, again, but with the following externally applied forces acting on each collision partner throughout the collision duration (with μ_1 = -0.9 and μ_2 = 0.3):

$$\begin{aligned} \mathbf{F}_{ext1} &= -\mu_1 m_1 g + 0.1t \\ \mathbf{F}_{ext2} &= \mu_2 m_2 g - 0.05t \end{aligned} \qquad (99).$$

The loading phase displacement response for the collision partners and the common collision interface are:

$$\begin{aligned} \mathbf{u}_1(t) &= -4.00\cdot 10^{-2} + 8.85t - \\ &\quad 1.13t^2 + 2.45\cdot 10^{-6} t^3 \\ &\quad +4.00\cdot 10^{-2}\cos(12.8t) \\ &\quad +0.480\sin(12.8t) \\ \mathbf{u}_2(t) &= 3.16\cdot 10^{-2} + 8.85t - \\ &\quad 1.13t^2 + 2.45\cdot 10^{-6} t^3 \\ &\quad -3.16\cdot 10^{-2}\cos(12.8t) \\ &\quad -0.379\sin(12.8t) \\ \mathbf{u}_{IV}(t) &= -2.03\cdot 10^{-2} + 8.85t - \\ &\quad 1.13t^2 + 2.45\cdot 10^{-6} t^3 \\ &\quad +2.03\cdot 10^{-2}\cos(12.8t) \\ &\quad +0.243\sin(12.8t) \end{aligned} \qquad (100).$$

The velocity and acceleration solutions are obtained, as before, by the simple first and second time derivatives. Closure terminates at 0.116 sec with a resultant peak displacement of 1.45 m, 0.663 m and 1.24 m, respectively for the collision partners and the common collision interface. The common velocity at closure is 8.59 msec^{-1}. The structural deflection experienced by each collision partner, during closure, is:

$$\begin{aligned} \boldsymbol{\delta}_1(t) &= -2.03\cdot 10^{-2} + 8.85t - \\ &\quad 1.13t^2 + 2.45\cdot 10^{-6} t^3 + \\ &\quad 2.03\cdot 10^{-2}\cos(12.8t) \\ &\quad +0.243\sin(12.8t) \\ \boldsymbol{\delta}_2(t) &= -1.97\cdot 10^{-2} + 8.85t - \\ &\quad 1.13t^2 + 2.45\cdot 10^{-6} t^3 + \\ &\quad 2.03\cdot 10^{-2}\cos(12.8t) \\ &\quad +0.243\sin(12.8t) \end{aligned} \qquad (101).$$

The peak deflection for each collision partner is 0.217 m and 0.572m, respectively. The peak collision force is 1.09 · 10^6 N. The time at which separation occurs is $t^* = 3.49\cdot 10^{-2}$ sec. The unload displacement response of the collision partners and the common collision interface are:

$$
\begin{aligned}
\mathbf{u}_1\left(t^*\right) &= 1.38 + 8.59t^* - \\
&\quad 1.13t^{*2} + 2.45\cdot 10^{-6}\, t^{*3} \\
&\quad +4.00\cdot 10^{-2}\cos\left(12.8t^*\right) \\
&\quad +3.24\cdot 10^{-2}\cos\left(47.3t^*\right) \\
&\quad -2.47\cdot 10^{-8}\sin\left(12.8t^*\right) \\
\mathbf{u}_2\left(t^*\right) &= 0.72 + 8.59t^* - \\
&\quad 1.13t^{*2} + 2.45\cdot 10^{-6}\, t^{*3} \\
&\quad -3.16\cdot 10^{-2}\cos\left(12.8t^*\right) \\
&\quad -2.56\cdot 10^{-2}\cos\left(47.3t^*\right) \\
&\quad +1.95\cdot 10^{-8}\sin\left(12.8t^*\right) \\
\mathbf{u}_2\left(t^*\right) &= 1.25 + 8.59t^* - \\
&\quad 1.13t^{*2} - 2.45\cdot 10^{-6}\, t^{*3} \\
&\quad -4.74\cdot 10^{-3}\cos\left(12.8t^*\right) \\
&\quad -3.84\cdot 10^{-3}\cos\left(47.3t^*\right) \\
&\quad +2.95\cdot 10^{-9}\sin\left(12.8t^*\right)
\end{aligned}
\qquad (102).
$$

The displacements at separation for the collision partners and the common collision interface are 1.72 m, 0.994 m and 1.54 m, respectively. The residual deflections for each collision partner are 0.174 m and 0.546 m with the corresponding deflection-time histories being:

$$
\begin{aligned}
\boldsymbol{\delta}_1\left(t^*\right) &= 0.137 + 3.53\cdot 10^{-7}\, t^* \\
&\quad -2.22\cdot 10^{-16}\, t^{*2} + 4.48\cdot 10^{-2}\cos\left(12.8t^*\right) \\
&\quad +3.62\cdot 10^{-2}\cos\left(47.3t^*\right) \\
&\quad -2.76\cdot 10^{-8}\sin\left(12.8t^*\right) \\
\boldsymbol{\delta}_2\left(t^*\right) &= 0.524 + 2.12\cdot 10^{-7}\, t^* \\
&\quad +2.69\cdot 10^{-2}\cos\left(12.8t^*\right) \\
&\quad +2.17\cdot 10^{-2}\cos\left(47.3t^*\right) \\
&\quad -1.66\cdot 10^{-8}\sin\left(12.8t^*\right)
\end{aligned}
\qquad (103).
$$

The kinematic response is shown in Figure 5.

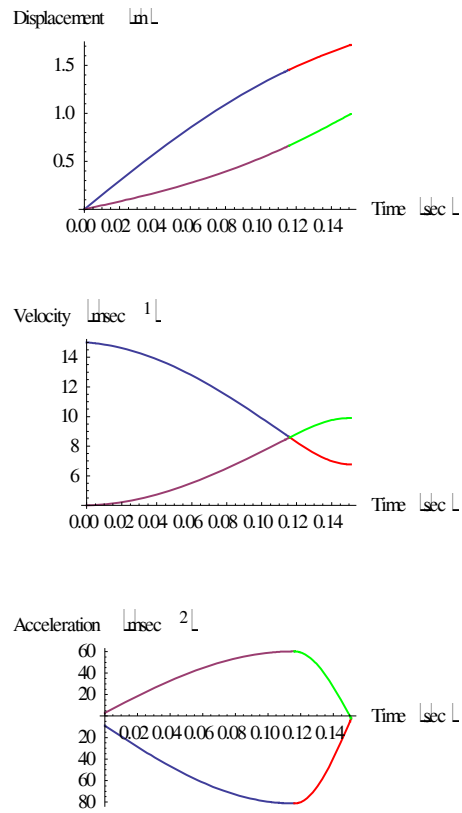

Figure 5. Kinematic response for the subject example with net unbalanced externally applied loads.

The force-deflection response follows that shown in Figure 4, but with the terminus of closure occurring at a lower peak collision force magnitude and with correspondingly lower magnitude peak and residual deflection magnitudes.

DISCUSSION

Net unbalanced externally applied loads, with respect to the system comprised of collision partners, are generally neglected when considering a collision phase analysis. This approach is generally apt in that the collision force magnitude and its impulse are generally the dominant factors in most collisions except for those that are classifiable as exceedingly minor severity collisions. Higher severity collisions involving a substantively disabled collision partner or collisions in which significant snagging is present during the collision phase represent exceptions to this generality. The typical case, however, involves collisions in which the net unbalanced externally applied loads, consisting of tire forces acting upon the collision partners during the impact, are of sufficient magnitude as to warrant their effects. The

approach taken in the subject study, in regards to the inclusion of such forces, was through the use of the Laplace transform of the second order coupled differential equations of motion. In this regard, two fundamental limitations are present. The first is that the collision partner structural modeling approach, for both closure and separation, is predicated on the linear elastic model. This approach, while allowing for the relatively simple decomposition of the undamped equations of motion, would require further modification in cases in which proportional damping is included in the structural response model. The linear load-linear unload model, while limited, can be utilized in certain appropriate cases [11]. The second limitation is that the set of externally applied loads must have a determinate Laplace transform and must not alter the dynamic stiffness matrix, $\mathbf{D}(s)$. Correspondingly, the framework developed herein can readily handle the temporally diminishing, operator-mediated, brake force that arises from the occupant kinematic response for a vehicle impacted in the rear and with the foot of the operator initially depressing the brake pedal if such force is defined on a purely temporal basis. However, if such an unbalanced load is defined in terms of struck vehicle forward displacement, the dynamic stiffness matrix will be altered. The worked example, while limited to the first two terms of a polynomial representation of the net externally applied force model, readily shows the utility of the subject modeling methodology. Future work will focus on expanding the subject model for modeling chain-reaction collinear collisions and upon including the effects of proportional damping with respect to the structural response model.

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