TRANSFORMING ANATOMICALLY ACQUIRED KINEMATIC PARAMETERS TO INERTIALLY REFERENCED COORDINATES

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Assume that some system of anatomically mounted inertial transducers has monitored the displacement of the anatomical body from some known initial position, orientation, and velocity. Assume further that the transducer outputs have been reduced to obtain the time history of the body translational acceleration expressed in some body referenced coordinate system and the time history of either the body angular velocity or angular acceleration expressed in the same body coordinates. Obtain the inertially referenced body position and orientation as well as the first and second time derivatives of these quantities.

These inertial parameters may be obtained in the manner diagrammed in figure 1. Body oriented angular acceleration is integrated to obtain body oriented angular velocity. The time history of the angular velocity vector is treated to obtain a time history of the quaternions of the body orientation. These quaternions yield the direction cosine matrix and the Euler angles relating the body axes to the inertially referenced coordinates.

These direction cosines are now applied to the body oriented angular acceleration, angular velocity, and translational acceleration to obtain their inertially referenced values. Finally, the inertially referenced translational acceleration as integrated to obtain the translational velocity and the body position.

The legality of the first step in this procedure, integration of body oriented angular cceleration to obtain body oriented angular velocity, depends on the following statement

$$\underline{\underline{C}}^t \frac{d(\underline{\omega})}{dt} = \frac{d(\underline{\underline{C}}^t \underline{\omega})}{dt}$$

which proceeds as

$$\underline{\underline{C}}^t \frac{d(\underline{\omega})}{dt} = \underline{\underline{C}}^t \frac{d(\underline{\omega})}{dt} + \frac{d\,\underline{\underline{C}}^t}{dt} \underline{\omega}$$

$$= \underline{\underline{C}}^t \frac{d(\underline{\omega})}{dt} - \underline{\underline{C}}^t (\underline{\omega} \times \underline{\omega})$$

$$= \underline{\underline{C}}^t \frac{d(\underline{\omega})}{dt}$$

In the next step the body oriented angular velocity is treated to obtain the quaternions of the body orientation. The quaternions are essentially a four parameter representation of the relation between the body axes and the inertially referenced system, the same quantity as that represented by the nine parameters of the direction cosines or the three parameters of the Euler angles. The advantage of the quaternions over the direction cosines is that where the direction cosines are bound by six constraints, the quaternions are bound only by one. The advantage over the Euler angle representation is that the quaternions have no such singularity as "gimbal lock," a differential change in orientation produces a differential change in the quaternions. A description of these quaternions and the manipulations used to obtain them is provided in the appendix.

Once the quaternions have yielded the direction cosines the rest of the parameters can be obtained by means of straightforward integration and matrix arithmetic.

List of Variables

$\underline{A}$	Inertially Referenced Acceleration
$\underline{A}'$	Body Referenced Acceleration
$\underline{\underline{C}}$	Direction Cosine Matrix Relating Inertial and Body Axes
$\underline{\underline{C}}_o$	Initial Value of the Direction Cosine Matrix
E	TheEuler Angles
$\underline{V}$	Inertially Referenced Velocity
$\underline{V}_o$	Initial Velocity
$\underline{X}$	Inertially Referenced Position
$\underline{X}'$	Initial Position
$\underline{\underline{q}}$	Quaternion
	Vector Part of the Quaternion
$\underline{\gamma}$	Inertially Referenced Angular Acceleration
$\underline{\gamma}'$	Body Referenced Angular Acceleration
$\underline{\omega}$	Inertially Referenced Angular Velocity
$\underline{\omega}'$	Body Referenced Angular Velocity
$\underline{\omega}_o$	Initial Angular Velocity

Notes on Quaternions

A quaternion consists of four parameters, these are a scalar, q_4, and a three dimensional vector $\underline{q}$. Multiplication of these quaternions is as follows:

$$\underline{\underline{q}}_1 * \underline{\underline{q}}_2 = \underline{\underline{q}}_3$$

where $q_{34} = q_{14} q_{24} - q_1 \; q_2$

$$\underline{q}_3 = q_{14} \underline{q}_2 + q_{24} \underline{q}_1 + \underline{q}_1 \times q_2$$

note that $\underline{\underline{q}}_1 * \underline{\underline{q}}_2 \neq \underline{\underline{q}}_2 * \underline{\underline{q}}_1$

It can be shown that this operation is associative:

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2) * \underline{\underline{q}}_3 = \underline{\underline{q}}_4$$

$$q_{44} = q_{34} \; (q_{14} q_{24} - \underline{q}_1 \bullet \underline{q}_2)$$

$$- \underline{q}_3 \quad (q_{14} \underline{q}_2 + q_{24} \underline{q}_1 + \underline{q}_1 \times \underline{q}_2)$$

$$\underline{q}_4 = (q_{14} q_{24} - \underline{q}_1 \bullet \underline{q}_2) \underline{q}_3 + q_{34} (q_{14} \underline{q}_2 + q_{24} \underline{q}_1 + \underline{q}_1 \times \underline{q}_2)$$

$$+ (q_{14} \underline{q}_2 + q_{24} \underline{q}_1 + \underline{q}_1 \times \underline{q}_2) \times \underline{q}_3$$

Since $(\underline{q}_1 \times \underline{q}_2) \bullet \underline{q}_3 = \underline{q}_1 \bullet (\underline{q}_2 \times \underline{q}_3)$

and $(\underline{q}_1 \times \underline{q}_2) \times \underline{q}_3 - (\underline{q}_1 \bullet \underline{q}_2) \underline{q}_3$

$$= \underline{q}_1 \times (\underline{q}_2 \times \underline{q}_3) - (\underline{q}_3 \bullet \underline{q}_2) \underline{q}_1$$

the expressions for $\underline{\underline{q}}_4$ become

$$_4 = q_{14}(q_{24}\, q_{34} - \underline{q}_2 \bullet \underline{q}_3)$$

$$- \underline{q}_1 \bullet (q_{34}\, \underline{q}_2 + q_{24}\, \underline{q}_3 + \underline{q}_2 \times \underline{q}_3)$$

$$\underline{q}_4 = q_{14}\,(q_{24}\, \underline{q}_3 + q_{34}\, \underline{q}_2 + \underline{q}_2 \times \underline{q}_3) + (q_{24}\, q_{34} - \underline{q}_2 \bullet \underline{q}_3)\, \underline{q}_1$$

$$+ \underline{q}_1 \times (q_{24}\, \underline{q}_3 + q_{34}\, \underline{q}_2 + \underline{q}_2 \times \underline{q}_3)$$

OR

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2) * \underline{\underline{q}}_3 = \underline{\underline{q}}_4 = \underline{\underline{q}}_1 * (\underline{\underline{q}}_2 * \underline{\underline{q}}_3)$$

The quaternion inverse is defined by

$$\underline{\underline{q}}_1^{-1} \Rightarrow q_{14} \Big/ \left|\underline{\underline{q}}_1\right|^2$$

$$- \underline{q}_1 \Big/ \left|\underline{\underline{q}}_1\right|^2$$

$$\underline{\underline{q}}_1 * \underline{\underline{q}}_1^{-1} = \underline{\underline{q}}_o$$

where

$$q_{o4} = (q_{14}^2 + \underline{q}_1 \bullet \underline{q}_1) \Big/ \left|\underline{\underline{q}}_1\right|^2 = 1$$

$$\underline{q}_o = (q_{14}\,(q_1 - q_1) - q_1 \times q_1) \Big/ \left|q_1\right|^2 = 0$$

$$= \underline{\underline{q}}_1^{-1} * \underline{\underline{q}}$$

This $\underline{\underline{q}}_o$ has the following property

$$\underline{\underline{q}}_o * \underline{\underline{q}}_1 = \underline{\underline{q}}_1 * \underline{\underline{q}}_o = \underline{\underline{q}}_1$$

The inverse of a quaternion product is

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2)^{-1} * (\underline{\underline{q}}_1 * \underline{\underline{q}}_2) = \underline{\underline{q}}_o$$

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2)^{-1} * \underline{\underline{q}}_1 * \underline{\underline{q}}_2 * \underline{\underline{q}}_2^{-1} * q_1^{-1} = q_2^{-1} * q_1^{-1}$$

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2)^{-1} * q_1 * q_1^{-1} = q_2^{-1} * q_1^{-1}$$

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2)^{-1} = \underline{\underline{q}}_2^{-1} * \underline{\underline{q}}_1^{-1}$$

Quaternions are normalized by

$$\underline{\underline{q}} = \left(\sum_{i=1}^{4} q_i^2\right)^{1/2}$$

Therefore

$$\begin{aligned}
(\underline{\underline{q}}_1 * \underline{\underline{q}}_2) = \Big[& q_{14}^2 \, q_{14}^2 - 2 q_{14} q_{24} \, (\underline{q}_1 \, \underline{q}_2) \\
& + (\underline{q}_1 \bullet \underline{q}_2)^2 + q_{14}^2 (\underline{q}_2)^2 + q_{24}^2 (\underline{q}_1)^2 \\
& + 2 q_{14} \, q_{24} (\underline{q}_1 \bullet \underline{q}_2) \\
& + 2 (q_{14} \underline{q}_2 + q_{24} \underline{q}_1) \bullet (\underline{q}_1 \times \underline{q}_2) \\
& + (\underline{q}_1 \times \underline{q}_2)^2 \Big]^{1/2} \\
= \Big[& q_{14}^2 q_{24}^2 + q_{14}^2 (\underline{q}_2)^2 + q_{24}^2 (\underline{q}_1)^2 + (\underline{q}_1 \bullet \underline{q}_2)^2 \\
& + (\underline{q}_1 \times \underline{q}_2)^2 \Big]^{1/2} \\
= \Big[& \left| \underline{\underline{q}}_1 \quad \underline{\underline{q}}_2 \right|^2 \Big]^{1/2} \\
= & \left| \underline{\underline{q}}_1 \right| \quad \left| \underline{\underline{q}}_2 \right|
\end{aligned}$$

The product

$$\underline{\underline{q}}_1 * \underline{\underline{q}}_2 * \underline{\underline{q}}_1^{-1} = \underline{\underline{q}}_3$$

yields

$$q_{34} = (q_{14}^2 \, q_{24} - q_{14} \, \underline{q}_1 \bullet \underline{q}_2$$

$$+ \underline{q}_1 \bullet (q_{14} \, \underline{q}_2 + q_{24} \, \underline{q}_1 + \underline{q}_1 \times \underline{q}_2)) / \left| \underline{\underline{q}}_1 \right|^2$$

$$= q_{24} \, (q_{14}^2 + \underline{q}_1 \bullet \underline{q}_1) / \left| \underline{\underline{q}}_1 \right|^2 = q_{24}$$

$$\underline{q}_3 = \Big[q_{14}^2 \, \underline{q}_2 + q_{14} \, q_{24} \, \underline{q}_1 + q_{14} \, (\underline{q}_1 \times \underline{q}_2)$$

$$- q_{14} \, q_{24} \, \underline{q}_1 + (\underline{q}_1 \bullet \underline{q}_2) \, \underline{q}_1$$

$$- q_{14} \, (q_2 \times q_1) - q_{24} \, (\underline{q}_1 \times \underline{q}_1) - (\underline{q}_1 \times \underline{q}_2) \times \underline{q}_1 \Big] / \left| \underline{\underline{q}}_1 \right|^2$$

Let $\underline{q}_2 = \underline{q}_2' + \underline{q}_2''$

where q_2' is parallel to $\underline{q}_1$

and q_2'' is perpendicular to q_1

$$\underline{q}_3 = \Big[q_{14}^2 \, \underline{q}_2' + (\underline{q}_1)^2 \, q_2' + q_{14}^2 \, \underline{q}_2''$$

$$+ 2 q_{14} \, \underline{q}_1 \times \underline{q}_2'' - (\underline{q}_1^2) \, \underline{q}_2'' \Big] / \left| \underline{\underline{q}}_1 \right|^2$$

$$= \underline{q}_2' + \frac{q_{14}^2 - (q_1)^2}{\left| \underline{\underline{q}}_1 \right|^2} \, \underline{q}_2'' + 2 \, \frac{q_{14}}{\left| \underline{\underline{q}}_1 \right|^2} \, (\underline{q}_1 \times \underline{q}_2'')$$

Since these three terms are perpendicular, the component of $\underline{q}_3$ parallel to $\underline{q}_1$ is just

$$\underline{q}_3' = \underline{q}_2'$$

The magnitude of $\underline{q}_3''$ is

$$\left|\underline{q}_3''\right| = \left[\frac{(q_{14}^2 - (\underline{q}_1)^2)^2 - (2q_{14}\,\underline{q}_1)^2}{\left|\underline{\underline{q}}_1\right|^4}\right]^{1/2} \qquad \left|q_2''\right| \quad = \left|q_2''\right|$$

The angle between q_2'' and q_3'' is

$$\theta = \mathrm{Sin}^{-1}\left|\frac{2\,q_{14}\,\left|q_1\right|}{\left|q_1\right|^2}\right|$$

Thus, the effect of

$$q_1 * q_2 * \underline{\underline{q}}_1^{-1}$$

is that q_{24} remains unchanged while the vector $\underline{q}_2$ is rotated about $\underline{q}_1$ an angle of $\mathrm{Sin}^{-1}\left(\frac{2q_{14}\left|\underline{q}_1\right|}{\left|\underline{\underline{q}}_1\right|^2}\right)$

A rotation x about an axis $\underline{\eta}$; $\left|\underline{\eta}\right| = 1$ can be represented by the quaternion

$$q_{14} = \text{Cos}\left(\frac{x}{2}\right)$$

$$\underline{q}_1 = \text{Sin}\left(\frac{x}{2}\right)\underline{\eta}$$

Operating on a quaternion made up of the vector to be rotated and to which any scalar may be appended as the quaternion scalar, a series of rotations becomes

$$\underline{\underline{q}}_n * \underline{\underline{q}}_{n-1} * \cdots \underline{\underline{q}}_2 * \underline{\underline{q}}_1$$

where $\underline{\eta}$ is given relative to fixed coordinates

OR

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_2 * \cdots \underline{\underline{q}}_{n-1} * \underline{\underline{q}}_n)$$

with $\underline{\eta}$ given relative to the rotated coordinates

Thus body oriented angular velocity has the quaternion form

$$\underline{\underline{q}}_\omega \Rightarrow q_{\omega 4} \simeq 1$$

$$\underline{q}_\omega \simeq (\underline{\omega}/2)\, dt$$

and the expression corresponding to the integration of angular velocity is

$$\underline{\underline{q}} = \prod_{t_i}^{t_f} \underline{\underline{q}}_\omega$$

Further Notes on Quaternions

Since the term $|\underline{\underline{q}}|^2$ turns up so frequently in the denominator of fractions it is required that this term be nonzero and positive. A useful value for this term is

$$\left|\underline{\underline{q}}\right|^2 = 1$$

Since the volume of computations will certainly produce deviations from unity, this term will be carried through in the calculation. This expression is the single constraint in the quaternion representation of rotation.

If $q_{24} = -q_{14}$

and $\underline{q}_2 = -\underline{q}_1$

then

$$(\underline{\underline{q}}_2 * \underline{\underline{q}}_3 * \underline{\underline{q}}_2^{-1}) = (\underline{\underline{q}}_1 * \underline{\underline{q}}_3 * \underline{\underline{q}}_1^{-1})$$

Any rotation in three dimensions can be represented by two distinct quaternions. However, since each quaternion corresponds only to a single rotation in three dimensions no difficulty is encountered.

$$(\underline{\underline{q}}_1 * \underline{\underline{q}}_3 * \underline{\underline{q}}_1^{-1}) \Rightarrow$$

$$\frac{1}{|\underline{\underline{q}}_1|^2} \begin{vmatrix} q_{14}^2 + q_{11}^2 - q_{12}^2 - q_{13}^2 & 2(q_{11}q_{12} - q_{13}q_{14}) & 2(q_{11}q_{13} + q_{14}q_{12}) \\ 2(q_{11}q_{12} + q_{13}q_{14}) & q_{14}^2 - q_{11}^2 + q_{12}^2 - q_{13}^2 & 2(q_{12}q_{13} - q_{11}q_{14}) \\ 2(q_{11}q_{13} - q_{12}q_{14}) & 2(q_{12}q_{13} + q_{11}q_{14}) & q_{14}^2 - q_{11}^2 - q_{12}^2 + q_{13}^2 \end{vmatrix} \underline{q}_3$$

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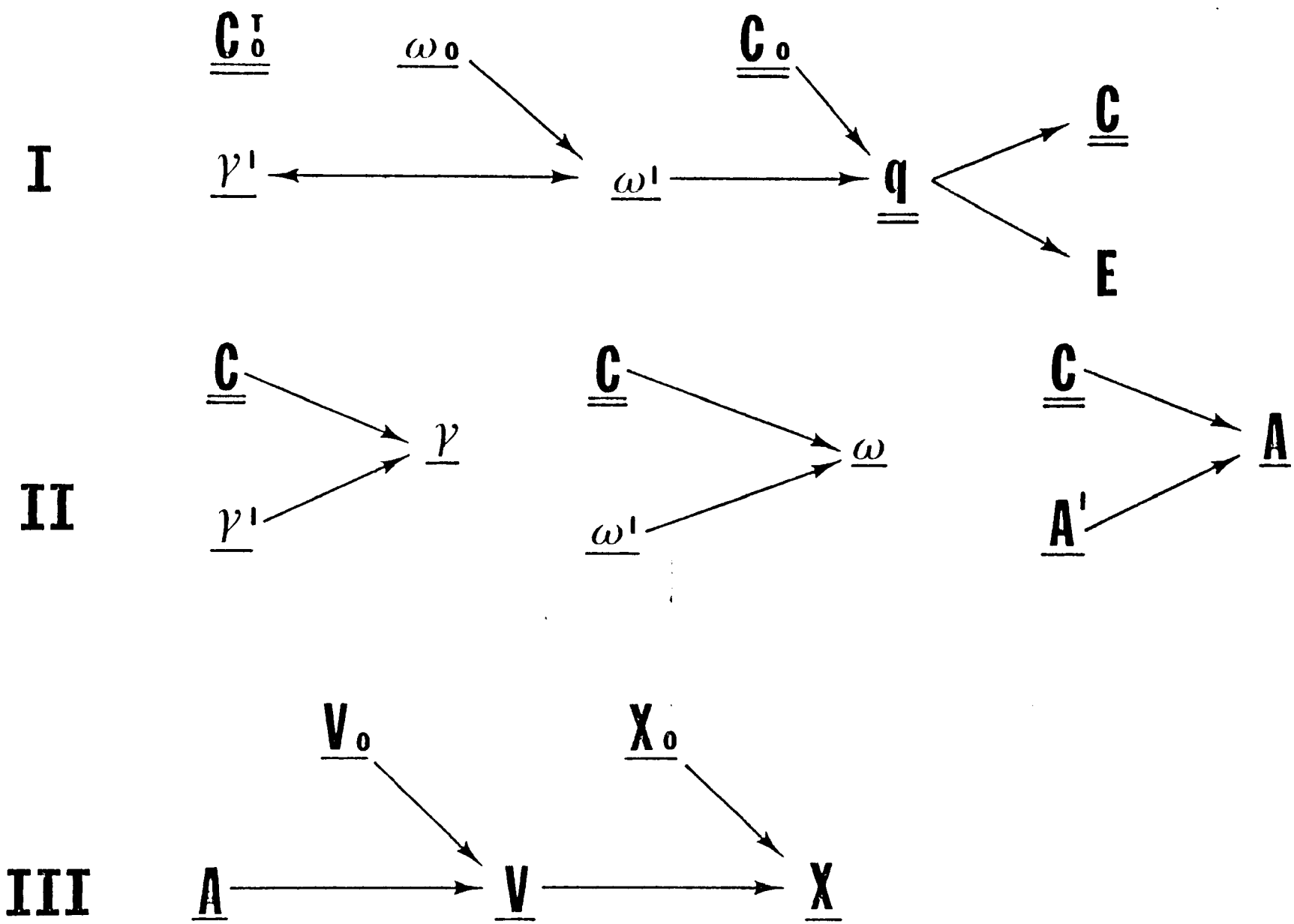

GENERATING INERTIALLY REFERENCED KINEMATIC PARAMETERS